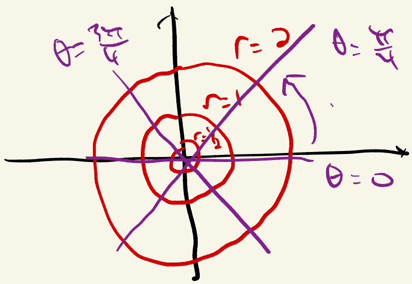
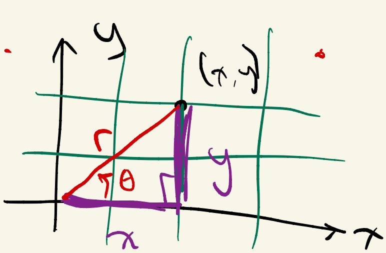
8-6 | Last time...

"Grid" in polar:

- r are distance to origin
- θ are rotations from x-axis





$\frac{y}{x} = \tan(\theta)$

$\theta = \arctan\left(\frac{y}{x}\right)$

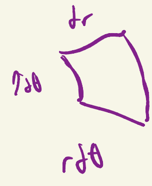
From trig:

$$\Rightarrow \boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \quad \text{(pythagorean)} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\iint_R f \, dA = \iint_R f(r,\theta)\, r \, dr \, d\theta$$

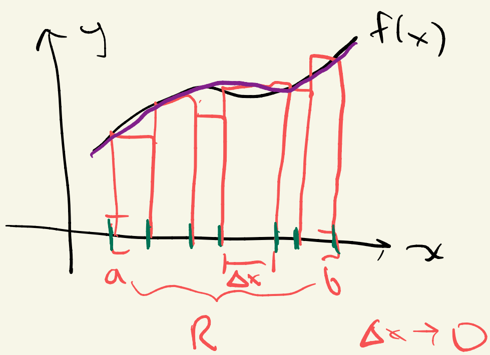
↑ don't forget



Today: Triple Integrals!

15.5

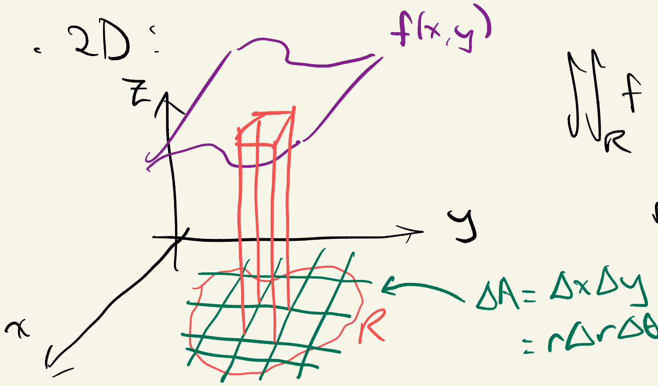
- In 1D:
  - Break down region R (a line segment [a, b]) into tiny line segments
    $\Rightarrow$ Riemann sum



- Area under curve

$$\int_R f(x)\,dx \approx \sum_{\substack{\text{tiny}\\ \text{rectangle}}} f(x)\,\Delta x$$

- 2D:



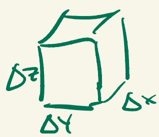
$$\iint_R f\,dA \approx \sum_{\substack{\text{thin}\\ \text{rectangular}\\ \text{prisms}}} f(x,y)\,\Delta A$$

rect. $\rightarrow$ $= \sum f(x,y)\,\Delta x \Delta y$

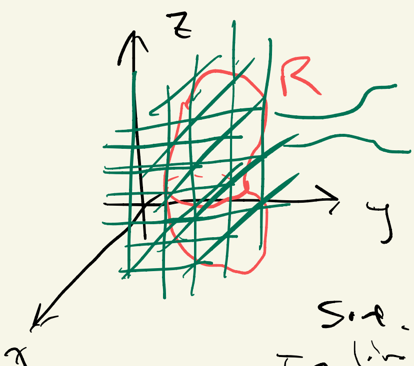
polar $\rightarrow$ $= \sum f(r,\theta)\, r\,\Delta r \Delta\theta$

- Volume under surface

- 3D



(can't draw -- f(x, y, z) requires 4 dimensions to draw x, y, z, and f(x,y,z))



$\Delta V = \Delta x \Delta y \Delta z$

$$\iiint_R f\,dV \approx \sum f(x,y,z)\,\Delta V$$

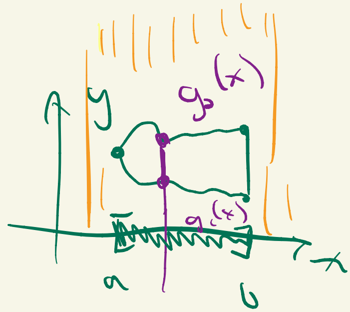
rect $\rightarrow$ $= \sum f(x,y,z)\,\Delta x \Delta y \Delta z$

Same business: partition R into boxes (ΔV)
In limit as box volume $\rightarrow 0$, Riemann sums should converge

Fubini still works. So can reorder our integral (take care of borders!)

How to perform integrals: Recall 2D

- Fix x (outer variable). How does y change as a function of x?

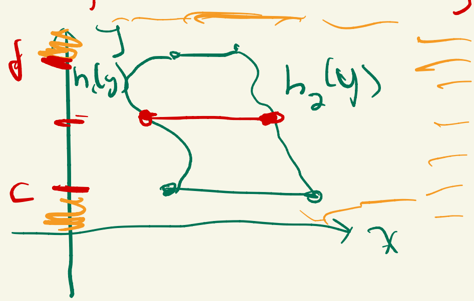


$$\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

outer bounds are like "shadow" of the graph

Or

- Fix y. How does x change as a function of y?



$$\int_{y=c}^{y=d}\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

"Shadow"

Now: 3D is no different.

- Fix (x,y). How does z change as a function of (x,y)?] inner integral

  outer variable — inner variable

- Then, where does (x,y) live? We'll use a "shadow" to find ~~this~~ region in xy plane
- Now — this a double integral! and we know how these work] Two outer 2 integrals

(Fubini still works. Could fix (y,z) first and find x bounds, or any other combo.)

Ex: Integrate the function 1 over the region $R \subseteq \mathbb{R}^3$ bounded by

$$z = f(x,y) = x^2 + 3y^2$$

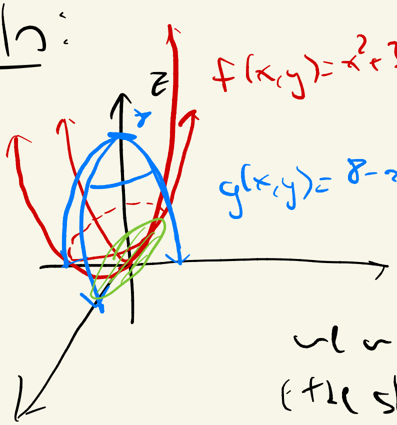
$$z = g(x,y) = 8 - x^2 - y^2$$

$8 - (x^2+y^2)$

paraboloid opening down

paraboloid opening up, with faster increasing direction

Soln:



we will need to find (x,y) region

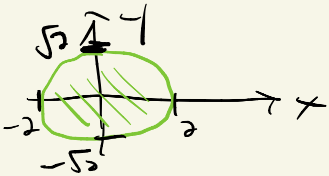
(the shadow thing)

Find first out this volume gets is our

$f(x,y) = g(x,y)$,

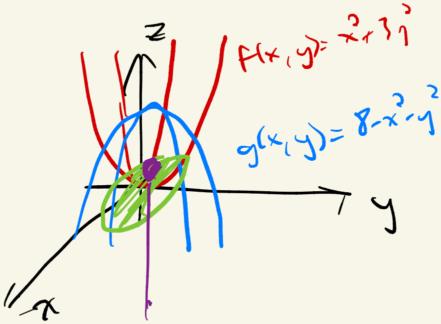
$x^2+3y^2 = 8-x^2-y^2 \Rightarrow \boxed{x^2+2y^2=4}$ Ellipse

Top down View:



$$x^2+2y^2=4$$

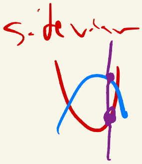
$$x^2+(\sqrt{2}y)^2=4$$



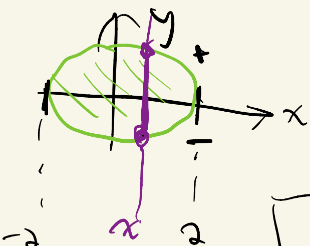
$\Rightarrow$ Fix (x,y). z

Inner integral

$$\int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz$$



Now, just a double integral for (x,y) region



$$x^2+2y^2=4 \Rightarrow 2y^2=4-x^2$$

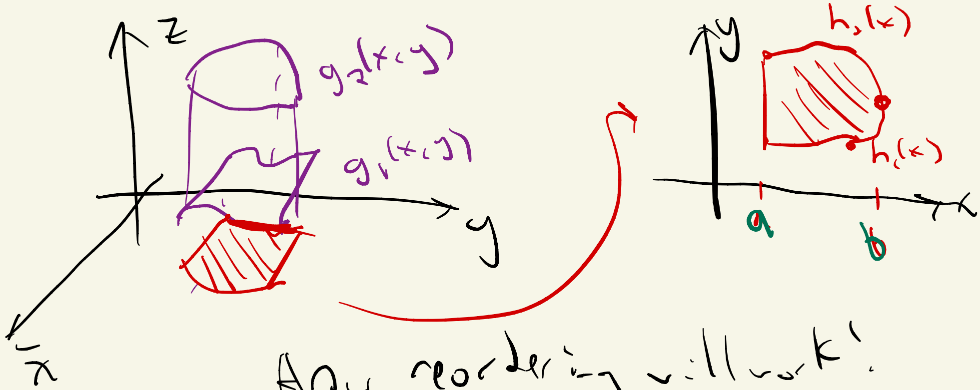
$$y=\pm\sqrt{\frac{4-x^2}{2}}$$

Fix x. How does y vary?

$$\boxed{\int_{x=-2}^{x=2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}}\int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz\,dy\,dx}$$

Fubini still works:

$$\iiint_R f(x,y,z)\,dV = \int_{x=a}^{x=b}\int_{y=h_1(x)}^{y=h_2(x)}\int_{z=g_1(x,y)}^{z=g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



Any reordering will work!

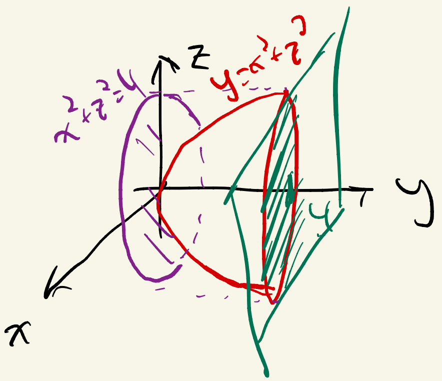
Just be careful with your functions.

10:57

Ex:

$$\iiint_R \sqrt{x^2+z^2}\,dV$$

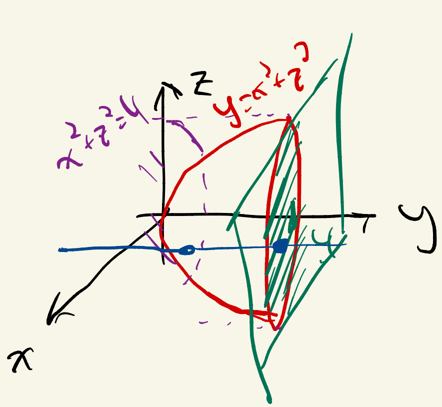
where R is region bounded by paraboloid $y=x^2+z^2$ and plane $y=4$



Shadow cast when parabola hits the plane:

$$4 = y = x^2+z^2$$

$$4 = x^2+z^2$$ ← circle in xz

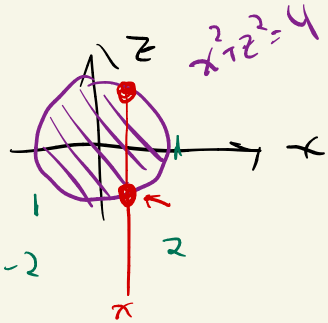


Fix (x,z). How is y a function of (x,z)?

$$\int_{y=x^2+z^2}^{y=4} \sqrt{x^2+y^2}\,dy$$

Now: Double integral over shadow region!
(in xz plane)

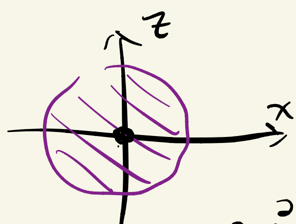
Fix x. How does z range?



$$\int_{x=-2}^{x=2}\int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}}\int_{y=x^2+z^2}^{y=4}\sqrt{x^2+y^2}\,dy\,dz\,dx$$

We'll evaluate this one.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\,dz\,dx$$



$r^2=x^2+z^2$

polar $= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\sqrt{x^2+z^2}\left(4-(x^2+z^2)\right)dz\,dx$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\sqrt{r^2}\,(4-r^2)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \sqrt{r^2}\,(4-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(4-r^2)\, r\,dr\,d\theta$$

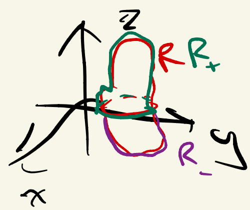
$$= \int_0^{2\pi}\int_0^2 4r^2 - r^4\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \frac{32}{3} - \frac{32}{5}\,d\theta$$

$$= 2\pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= \frac{128\pi}{15}$$



Triple integrals obey same rules:

1. Linear α a scalar, f, g functions

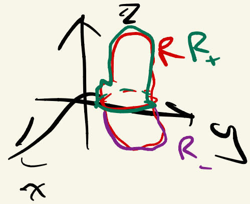
$$\iiint_R \alpha f(x,y,z) + g(x,y,z)\,dV = \alpha \iiint_R f(x,y,z)\,dV + \iiint_R g(x,y,z)\,dV$$

2. Domination:

If $f(x,y,z) \geq g(x,y,z)$ for every $(x,y,z) \in R$

$$\iiint_R f(x,y,z)\,dV \geq \iiint_R g(x,y,z)\,dV$$

3. If $R = R_+ \cup R_-$, and R_+ is disjoint from R_-



$$\iiint_R f(x,y,z)\,dV = \iiint_{R_+} f(x,y,z)\,dV + \iiint_{R_-} f(x,y,z)\,dV$$

4. <u>Fubini's Thm</u>

If f is continuous on closed region R, we can reorder dxdydz as needed

6 orders: $dx\,dy\,dz \quad dy\,dx\,dz \quad dz\,dx\,dy$
$dx\,dz\,dy \quad dy\,dz\,dx \quad dz\,dy\,dx$

5. Can calculate

$$\text{Volume}(R) = \iiint_R 1 \cdot dV, \quad \text{Ave}(f) = \frac{1}{\text{Vol}(R)} \iiint_R f(x,y,z)\,dV$$

(just like $\text{Area}(R) = \iint_R dA$)

15.6

Def:

- Average density of an object with mass M and Volume V is $\rho_{avg} = M/V$

If we calculate average density over infinitesimally small volumes, gives density function $\rho(x,y,z)$.

$$M = \iiint_R \rho(x,y,z)\, dV$$

$$\text{mass}\,[\text{kg}] = \underbrace{\iiint_R}_{\text{add many tiny volumes}} \text{density}\left[\frac{\text{kg}}{\text{m}^3}\right] \cdot dV\,[\text{m}^3]$$

$$\text{stuff} = \iiint_R \text{stuff density} \cdot dV$$

- charge (charge density)
- population (population density)
- # moles of chemical (concentration)
- power (power density)
- probability (probability density) ← Quantum mechanics

- Similarly for $\mathbb{R}^2$ and $\mathbb{R}^1$
  - $\mathbb{R}^2$: surface density
  - $\mathbb{R}^1$: density

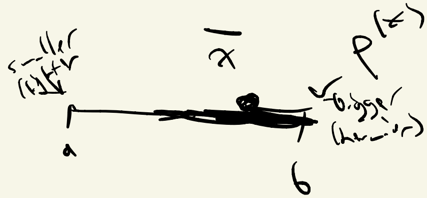
$$M = \iint_R \sigma(x,y)\, dA$$

Center of mass

"weighted average position":

$\mathbb{R}^1$

$$\bar{x} = \frac{1}{M}\int_{[a,b]} x\rho(x)\,dx$$



$\mathbb{R}^2$

$$M_y = \bar{x} = \frac{1}{m}\iint_R x\sigma(x,y)\,dA$$

$$M_x = \bar{y} = \frac{1}{m}\iint_R y\sigma(x,y)\,dA$$

"1st moment"

$\mathbb{R}^3$

$$M_{yz} = \bar{x} = \frac{1}{m}\iiint_R x\rho(x,y,z)\,dV$$

$$M_{xz} = \bar{y} = \frac{1}{m}\iiint_R y\rho(x,y,z)\,dV$$

$$M_{xy} = \bar{z} = \frac{1}{m}\iiint_R z\rho(x,y,z)\,dV$$

moments of planes

- In physics/engineering (probability), these integrals have special meaning. "first moments" allow calculation of center of mass. In probability, these give expected value

- Also important: second moments

Moment of inertia

will continue on monday